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# AGAIN ON PARETO MINIMUM POINTS OF AN INCONSISTENT SYSTEM

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## Introduction

Following[1] let consider the following system of $m$ equations and $n$ unknowns:

$$f_k(z) = \sum_{j=1}^{n} a_{kj}z - b_j = 0, \quad k \in M = \{1, 2, \ldots, m\} \tag{1}$$

or, equivalently,

$$Az - b = 0$$

where the notations are obvious:

$$A = (a^k)_{k=1,2,\ldots,m} := (a_{k1}, a_{k2} \ldots, a_{kn})_{k=1,2,\ldots,m} \in \mathrm{M}_{m,n}(\mathbb{C}), \quad b = (b_1, b_2, \ldots b_m) \in \mathrm{M}_{m,1}(\mathbb{C})$$

and $$z = (z_1, z_2, \ldots, z_n) \in \mathrm{M}_{n,1}(\mathbb{C}) \quad \text{(here } a^k := (a_{k1}, a_{k2}, \ldots, a_{kn})\text{)}.$$

**Definition 1.1.** *$z \in \mathbb{C}^n$ is an infrasolution of the system (1) if there is no $u \in \mathbb{C}^n$ so that:*

– $Au \neq Az$
– If, for $k \in M, f_k(z) = 0$, then $f_k(u) = 0$
– If, for $k \in M, f_k(z) \neq 0$, then $|f_k(u)| < |f_k(z)|$

Let us denote the set of all infrasolutions of (1) by I$S(A, b)$

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[1] I. Marusciac: *Infrapolynomials and Pareto optimization*, Mathematica (Cluj), 22 (45), (1980), No. 2, pp. 297–307; E.I. Remez: *Basis of the Numerical Methods in Tchebycheff's Approximation*, Izd. Kiev, 1969.

**Lemma 1.1.** *System (1) is consistent if and only if every solution* z *of the system is also an infrasolution,* **i.e.** I*S* (*A*, *b*) *coincides with the set of* <u>*all*</u> *solutions of (1).*

**Definition 1.2.** $z \in \mathsf{C}^n$ *is a Pareto minimum solution or Pareto minimum point of the system (1) if there is no* $u \in \mathsf{C}^n$ *such that:*

- $|f_k(u)| \leq |f_k(z)|$ for all $k \in M$ ;
- There is a $k_0 \in M$ so that $|f_{k_0}(u)| < |f_{k_0}(z)|$.

**Definition 1.3.** $z \in \mathsf{C}^n$ *is called a weak Pareto minimum solution of the system (1) if there is no* $u \in \mathsf{C}^n$ *such that* $|f_k(u)| < |f_k(z)|$ *for all* $k \in M$ .

We denote by P*A* (*A*, *b*), and, respectively by P$A^*$ (*A*, *b*) the sets of all Pareto minimum solutions, respectively for all weak Pareto minimum solutions of the system (1).

**Definition 1.4.** *An approximate solution* $z_0 \in \mathsf{C}^n$ *of the system (1) is called Tschebychev uniform best approximation solution of (1), or a Tschebychev's point for the system (1) if*

$$\max_{k \in M} \{ |f_k(z_0)| \} = \inf_{z \in \mathsf{C}^n} \max_{k \in M} \{ |f_k(z)| \} \tag{2}$$

**Definition 1.5.** *Let now* $(\lambda_k)^n$ *be a system of weights, so that* $\lambda_k > 0$, $\sum_{k=1}^n \lambda_k = 1$ *and let also* $p > 0$. *An approximate solution* $z^* \in \mathsf{C}^n$ *of the system (1) is called solution of the least deviaton from* 0 *in weighted mean of order p of (1), if*

$$\left( \sum_{k=1}^{n} \lambda_k \, |f_k(z^*)|^p \right)^{1/p} = \inf_{z \in \mathsf{C}^n} \left( \sum_{k=1}^{n} \lambda_k \, |f_k(z)|^p \right)^{1/p} \tag{3}$$

In a particular case when $p = 2$ and $\lambda_k = 1/n$ for all $k \in M$, the solution of the least deviation from 0 of the system (1) is called the least squares solution of the system (1).

**Definition 1.6.** *A matrix* $A \in \mathsf{M}_{m,n}(\mathsf{C})$, $m \geq n$ *is said to have the "H-property" (Haar property) if all quadratic submatrices of A of order n have exactly rank n.*

**Lemma 1.2.** *If* $z_0 \in \mathsf{C}^n$ *and* $A \in \mathsf{M}_{m,n}$, $m \geq n$ *and A has the H-property and if there exist* $l \geq n$ *and* $k_1, k_2, \ldots, k_l \in M$ *so that*

$$f_{k_j}(z_0) = 0, \qquad for \quad j = 1,2,\ldots,l,$$

then $z_0 \in \mathrm{P}A(A,b)$.

**Definition 1.7.** *Let us denote by* I$S_0$ (*A*, *b*), $\mathrm{P}A_0(A,b)$ *and, respectively* $\mathrm{P}A_0^*(A,b)$ *the subsets of infrasolutions, Pareto minimum solutions, respectively weak Pareto minimum solutions z of system (1), for which* $f_k(z) \neq 0$ *for all* $k \in M$.

**Theorem 1.1.** *Between the three classes defined above, we have the following inclusions:*

$$\mathrm{P}A(A,b) \subset \mathrm{I}S(A,b) \subset \mathrm{P}A^*(A,b).$$

**Definition 1.8.** *Let* $X \subset \mathbb{C}^n$ *and* $f : X \to R^m$, $g : X \to R^p$ and let $\Omega = \{x \in X : g(x) \le 0\} \ne \Phi$. $x_0$ *is called a Pareto minimum point of f on* $\Omega$ *(or Pareto minimum solution) of the problem:*

$$f(x) \to \min,$$

under the condition:

$$g(x) \le 0$$

if there is no $x \in \Omega$ such that

$$f(x) \le f(x_0), \qquad f(x) \ne f(x_0),$$

(Here, the inequalities mean inequalities between the similar **real** components of *f*, respectively *g*).

In [1] we proved some results concerning these classes. In what follows we will need the following theorems:

**Theorem 1.2.** *(* [1]*) If* $z^* \in \mathbb{C}^n$ *is a solution of the least deviation from* 0 *in weighted mean of order p of the system (1), then* $z^*$ *is a Pareto minimum solution of (1),* **i.e.** $z^* \in \mathrm{P}A(A,b)$.

**Theorem 1.3.** *(* [1]*) Let* $A \in \mathrm{M}_{m,n}(\mathbb{C})$ *and* $b \in \mathbb{C}^n$. *Then* $z_0 \in \mathrm{P}A_0(A,b)$ *if and only if there exist p functions* $f_{k_1}, f_{k_2}, \ldots, f_{k_p}$ $(2 \le p \le 2n+1)$ *and p positive numbers* $\lambda_\nu$ *with* $\sum_{nu=1}^{p} \lambda_\nu = 1$ *so that*

$$\sum_{\nu=1}^{p} \lambda_\nu \frac{a_{k_\nu j}}{f_{k_\nu}(z_0)} = 0, \qquad j \in \{1, 2, \ldots, n\}$$

If $z_0 \in \mathrm{I}S_0(A,b)$, then $z_0 \in \mathrm{I}S_0(B,c)$, where

$$B = \left( a^{k_1} a^{k_2} \ldots a^{k_p} \right)^t, \qquad c = (b_{k_1} b_{k_2} \ldots b_{k_p})^t$$

**Corolarry 1.1.** *(* [1]*) With the notations and conditions of* **Theorem 1.3***, if A has the “*$H$*-property”, then number p in* **Theorem 1.3** *satisfies the inequality* $n+1 \le p \le 2n+1$ *and if A is a matrix with* **real** *elements , then p = n +* 1.

**Main results**

**Lemma 2.1.** *Let A be a matrix of order mxn with complex elements. Let*

$$B = \left( a^{k_1} a^{k_2} \ldots a^{k_p} \right)^t$$

be a submatrix of *A* having the order $p\mathrm{x}n, p \ge n$. If *A* has the “*H*-property”, then $\mathrm{P}A(B,c) \subseteq \mathrm{P}A(A,b)$ $(c = (b_{k_1} b_{k_2} \ldots b_{k_p})^t)$.

Proof. Let $z_0 \in \mathrm{P}A(B,c)$ and assume that $z_0 \notin \mathrm{P}A(A,b)$. Then, we can find $u \in \mathbb{C}^n$ so that:

$$|f_k(u)| \leq |f_k(z_0)| \qquad \text{forall} k \in M$$

and it exists $k_0 \in M$ so that

$$|f_{k_0}(u)| < |f_{k_0}(z_0)| \tag{4}$$

From the first inequality we have that $|f_k(u)| \leq |f_k(z_0)|$ for all $k \in \{k_1, k_2, \ldots, k_p\}$. If there were a $k' \in \{k_1, k_2, \ldots, k_p\}$ such that $|f_{k'}(u)| < |f_{k'}(z_0)|$, $z_0$ would not be in $\mathrm{P}A(B,c)$ and this is a contradiction. It follows that for every $k \in \{k_1, k_2, \ldots, k_p\}$ we have $|f_k(u)| = |f_k(z_0)|$.

Let now consider $v = (u + z_0)/2$.

$$|f_k(v)| = |a^k v - b_k| = \left| \frac{a^k u + a^k z_0}{2} - b_k \right| = \frac{1}{2}\left|(a^k u - b_k) + (a^k z_0 - b_k)\right| \leq$$

$$\leq \frac{1}{2}\Big( |a^k u - b_k| + |a^k z_0 - b_k| \Big) \leq \frac{1}{2} 2\, |a^k z_0 - b_k| = f_k(z_0) \qquad \textit{for all} \, k \in M$$

It follows that $|f_k(v)| = |f_k(z_0)|$ for each $k \in \{k_1, k_2, \ldots, k_p\}$. This means that $(a^k)v = (a^k)z_0$ for all $k \in \{k_1, k_2, \ldots, k_p\}$. Because $p \geq n$ and $B$ has the "*H*-property", the last equalities show that $u = z_0$, contradicting (4). Hence, we have that $z_0 \in \mathrm{P}A(A,b)$.

**Theorem 2.1.** *Let $A \in \mathrm{M}_{m,n}(\mathbb{C})$ and $b \in \mathbb{C}^n$. (i)If $z_0 \in \mathrm{P}A_0(A,b)$, then we can find p functions $f_{k_1}, f_{k_2}, \ldots, f_{k_p}$ and p positive numbers $\nu_\lambda$ with $\sum_{\nu=1}^{p} = 1$ so that*

$$\sum_{\nu=1}^{p} \lambda_\nu \frac{a_{k_\nu j}}{f_{k_\nu}(z_0)} = 0, \qquad j \in \{1, 2, \ldots, n\} \tag{5}$$

(ii) If matrix $A$ has the "*H*-property", then conditions (5) are sufficient for $z_0 \in \mathrm{P}A_0(A,b)$ and $n + 1 \leq p \leq 2n + 1$.

(iii) If $A$ is like in (ii) and **real**, then $p = n + 1$.

Proof. The conclusions are simple applications of **Theorem 1.3** and also of **Lemma 2.1.**

**Remark 2.1.** *In the conditions of* **Theorem 2.1**, *we have:*

$$\sum_{\nu=1}^{p} \lambda_\nu \frac{b_{k_\nu}}{f_{k_\nu}(z_0)} = -1 \tag{6}$$

Proof. By multiplying (5) by $z_{0}, j$ and by adding term by term the corresponding relations ($j = 1, 2, \ldots, n$), we obtain:

$$\sum_{\nu=1}^{p} \lambda_\nu \frac{\sum_{j=1}^{n} a_{k_\nu j} z_{0,j}}{f_{k_\nu}(z_0)} = 0.$$

Relation (6) follows immediately from the above equality, taking into account that $\sum_{k=1}^{n} a_{k_\nu j} z_{0,j} = f_{k_\nu}(z_0) + b_k$ and $\sum_{\nu=1}^{p} \lambda_\nu = 1$.

Let us now make the following notations: consider (without any loss of generality) that indices $k_1, k_2, ..., k_p$ in **Theorem 2.1** are the first $p$ indices 1, 2, ..., $p$ and thus condition (5) becomes

$$\sum_{k=1}^{p} \lambda_k \frac{a_{kj}}{f_k(z_0)}, \ j = 1,2,\dots,n, \ \lambda_k > 0, \qquad \sum_{k=1}^{p} \lambda_k = 1 \tag{7}$$

Denoting $d_k = \lambda_k / \mid f_k(z_0) \mid^2$, system (6) can be written as:

$$\sum_{k=1}^{p} d_k f_k(z_0) \overline{a}_{k,j}, \qquad j = 1, 2, \dots, n \tag{8}$$

Let us now denote:

$$D_{jl} = \sum_{k=1}^{p} d_k a_{il} \overline{a}_{kj},$$

$$D_j = \sum_{d_k b_k \overline{a}_{kj}} .$$

We can rewrite the system (7) in the form:

$$D_{j1} z_1 + D_{j2} z_2 + \dots + D_{kn} z_n = D_j, \qquad j = 1, 2, \dots, n \tag{9}$$

Using the method in [3] we can easily write the solution of the system (8) in the following way:

**Theorem 2.2.** *Let (1) be an inconsistent system for which matrix A has the "H-property". An approximate solution* $z_0 \in \mathbb{C}^n$ *with* $f_k(z_0 \neq 0, k \in M$ *of this system is a Pareto minimal solution of (1) if and only if there exists a subsystem of (1) with p equations* $(n+1 \leq p \leq 2n+1)$ *(we assume that* $f_k(z) = 0, k = 1,2,\dots,p)$ *and there exist also p positive numbers* $d_k$ *such that:*

$$z_{0k} = \frac{\sum d_{k_1} d_{k_2} \dots d_{k_n} \overline{D}(k_1, k_2, \dots, k_n; A) D(k_1, k_2, \dots, k_n; A; b)}{\sum d_{k_1} d_{k_2} \dots d_{k_n} \mid D(k_1, k_2, \dots, k_n; A) \mid^2} \tag{10}$$

where:

$$D(k_1, k_2, \dots, k_n; A) := \begin{vmatrix} a_{k_1 1} & a_{k_1 2} & \cdots & a_{k_1 n} \\ a_{k_2 1} & a_{k_2 2} & \cdots & a_{k_2 n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k_n 1} & a_{k_n 2} & \cdots & a_{k_n n} \end{vmatrix},$$

and $D(k_1, k_2, \dots, k_n; A; b)$ is the determinant obtained from $D(k_1, k_2, \dots, k_n; A)$ by replacing the column $(a_{k_1 j} a_{k_2 j} \dots a_{k_n j})^t$ with $(b_{k_1} b_{k_2} \dots b_{k_n})^t$. $\overline{D}(k_1, k_2, \dots, k_n; A)$ is the complex conjugate of $D(k_1, k_2, \dots, k_n; A)$ and the sum $\Sigma$ is taken for all the values of $k_1, k_2, \dots, k_n$ in the set {1, 2, ..., $p$}.

If matrix $A$ is **real** (**i.e.** the system is real) and has the "$H$-property", by **Corolarry 1.1** we have that $p = n + 1$ and, hence, relations (10) becomes (for $j = 1, 2, ..., n$):

$$z_{0j} := x_{0j} = \frac{\sum_{k=1}^{n+1} d_1 d_2 \dots d_{n+1} D_k(A) D_{kj}(A; b)}{\sum_{k=1}^{n+1} d_1 \dots d_{k-1} d_{k+1} \dots d_{n+1} \mid D(A) \mid^2} \tag{11}$$

where

$$D_k(A) = \begin{vmatrix} a_{k_1 1} & a_{k_1 2} & \cdots & a_{k_1 n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k-1,1} & a_{k-1,2} & \cdots & a_{k-1,n} \\ a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \end{vmatrix}$$

and $D_{kj}(A; b)$ is obtained in the same way as above.

Equalities (10) can be written in a more simple form as:

$$z_{0j} = \sum \Lambda_{k_1 k_2 \dots k_n} \frac{D_j(k_1, k_2, \dots, k_n; A; b)}{D(k_1, k_2, \dots, k_n; A)}, \qquad j = 1, 2, \dots, n \tag{12}$$

and

$$\Lambda_{k_1 k_2 \dots k_n} = \frac{d_{k_1} d_{k_2} \dots d_{k_n} \mid D(k_1, k_2, \dots, k_n; A) \mid^2}{\sum d_{k_1} d_{k_2} \dots d_{k_n} \mid D(k_1, k_2, \dots, k_n; A) \mid^2}.$$

We see that $\Lambda_{k_1 k_2 \dots k_n} > 0$ and $\sum_{k=1}^{n+1} \Lambda_{k_1 k_2 \dots k_n} = 1$ and in the real case equalities (11) can be re-written as:

$$x_{0j}=\sum_{k=1}^{n+1}\Lambda_k\frac{D_{k,j}(A;b)}{D_k(A)},\qquad j=1,2,\ldots,n,$$

where $\Lambda_k>0$ and $\sum\Lambda_k=1$.

**Theorem 2.3** *Let (1) be an inconsistent system. If matrix A of the system has the "H-property", there exists a subsystem of (1),*

$$f_{k_j}(z)=0,\ j=1,2,\ldots,p\ \text{ and }\ n+1\le p\le 2n+1 \tag{13}$$

so that:

$$\mathrm{P}A_0(A,b)=conv\{z^c\},$$

where $conv\{z^c\}$ is the convex hull of all Cramer solutions $z_0$ of the $n$x$n$ subsystems of (13)

Proof. For $z_0\in\mathrm{P}A_0(A,b)$, if we apply **Theorem 2.2**, we conclude that relation (12) is true. Let $\Upsilon\in\mathbf{C}^n$, $\Upsilon=(\Upsilon_1,\Upsilon_2,\ldots,\Upsilon_n)$ be a Cramer solution of the system $f_{k_j}=0$ ($j=1, 2, ..., n$), **i.e.**:

$$\Upsilon_j=W_j(k_1,k_2,\ldots k_n;A;b):=\frac{D_j(k_1,k_2,\ldots,k_n;A;b)}{D(k_1,k_2,\ldots,k_n;A)},\qquad j=1,2,\ldots,n.$$

System $f_{k_j}=0$ ($j=1, 2, ..., n$) is a subsystem of system (13) and we have also:

$$z_{0j}=\sum\Lambda_{k_1,k_2,\ldots,k_n}W_j(k_1,k_2,\ldots k_n;A;b),$$

so that

$$z_0=\sum\Lambda_{k_1,k_2,\ldots,k_n}W(k_1,k_2,\ldots k_n;A;b),$$

where

$$W(k_1,k_2,\ldots k_n;A;b):=$$
$$\big(W_1(k_1,k_2,\ldots k_n;A;b),W_2(k_1,k_2,\ldots k_n;A;b),\ldots,W_n(k_1,k_2,\ldots k_n;A;b)\big),$$

and thus, $z_0\in conv\{z^c\}$. Conversely, if $z_0\in conv\{z^c\}$, then $z_0$ is an open convex combination of Cramer solutions $W(k_1,k_2,\ldots k_n;A;b)$ of subsystems having the form $f_{k_j}=0$ ($j=1, 2, ..., n$). Consequently, (12) holds, and thus, by **Theorem 2.2**, $z_0\in\mathrm{P}A_0(A,b)$.

In [1] we saw (see also **Theorem 1.2** in the first section of this article) that the least square solution and the best approximate solution of the system (1) (with *A* having the "*H*-property") are in fact two special cases of the Pareto minimal solutions of the same system. In other terms, using the notations defined above, this means that for a certain choice of weights $\Lambda_{k_1,k_2,\ldots,k_n}$ in formula (12) we can obtain the two above mentioned approximate solutions. We will look now at the least square solutions of system (1), which characterize them.

Obtaining the least square solution of system (1) means minimization of function

$$\mathsf{F}(f)=\sum_{k=1}^{m} f_k(z)^2=\sum_{k=1}^{m} f_k(z)\bar{f}_k(z)\,.$$

Hence, here we have to solve system:

$$\frac{\partial f}{\partial \bar{z}_j} = 2\sum_{k=1}^{m} f_k(z)\bar{a}_{kj} = 0, \qquad j = 1, 2, \ldots, n \tag{14}$$

But system (14) is similar to (8) ($p = m$ and instead of $z_0$ we have here $z$) with $d_1 = d_2 = \ldots = d_m = 1$. We have proved the following result:

**Theorem 2.4.** *If matrix A of system (1) has the "H-property", then* $z_0 \in \mathrm{P}A_0(A,b)$ *is the least square solution for system (1) if and only if* $d_1 = d_2 = \ldots = d_m = 1$ *in relation (8), which means that:*

$$z_{0j} = \frac{\overline{D}(k_1,k_2,\ldots,k_n;A)D_j(k_1,k_2,\ldots,k_n;A;b)}{|\,D(k_1,k_2,\ldots,k_n;A)\,|^2}, \qquad j = 1, 2, \ldots, n \tag{15}$$

In a similar way it is possible to characterize the best approximation solution of system (1). Formula (15) is a synthetic form of the least square solution of an inconsistent system.

#### References

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#### Summary

Starting from the previous papers G. Caristi (*Pareto solutions of an inconsistent system*, Applied Mathematical Sciences, Vol. 4, 2010, No. 10) and M. Fekete, J.L. Walsh (*On restricted infrapolynomials*, Proceedings of the National Academy of Sciences, U.S.A., 37 (1951)), in this paper we

show some new results and remarks concerning the last square solutions and we consider the best approximation of the solution of an inconsistent system[2].

## PONOWNIE NA TEMAT PARETO OPTYMALNYCH PRZYBLIŻEŃ ROZWIĄZAŃ UKŁADÓW SPRZECZNYCH


### Streszczenie

W niniejszym artykule, opierając się na pracach G. Caristi (*Pareto solutions of an inconsistent system*, Applied Mathematical Sciences, Vol. 4, 2010, No. 10) i M. Fekete, JL Walsh (*On restricted infrapolynomials*, Proceedings of the National Academy of Sciences, U.S.A., 37 (1951)), pokazano nowe wyniki i uwagi dotyczące metody najmniejszych kwadratów oraz najlepsze, według autorów, przybliżenie rozwiązania sprzecznego układu.


[2] AMS-2000 Mathematics Subject Classification: 49K10.